ADVANCED PROPERTIES OF NONLOCAL OPERATORS AND APPLICATIONS

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1. The research project

1.1. **State of the play.** Fractional operators have been intensively studied since nineteenth century, when the fractional calculus has been introduced. Indeed, the first trace of this interest already appears at the beginning of the calculus history with Leibnitz, Bernoulli, De L'Hôpital and then with Euler and Laplace. However, with Liouville and Abel work the scientific interest grew up, because after their contribute it was possible to solve the tautochrone problem.

Important improvements came from Riemann and then from Marchaud, Weyl, Grünwald and Letnikow, few comments about this historic part may be found in [10], until to arrive to Marcel Riesz and Frostam who introduced a potential theory associated with kernel like $\|x\|^{\alpha-n}$, $\alpha \in (0,2)$, where n is the dimension of the Euclidean space. A collection of these results may be found in [9]. In any case, a brief historical introduction about this subject may be read in [20], see also [15].

In 2006 L. Caffarelli and L. Silvestre gave a new impulse to this research, because they put in evidence that some nonlocal problems may be reduced to local problems applying the extension method described in [3], see for instance the application to the thin obstacle in [4] and [7] for a revisitation of Sobolev spaces with respect to the fractional Laplace operator as well as the papers [1] and [2] for finding other applications of the extension method to Marchaud derivative.

Concerning non-commutative structures, very few results in literature exist. In particular, we remark [12], [14], [13] , [11] and more recently also in the paper [16]. Very few results are available for functions that possibly diverge.

1.2. **Nonlocal operators.** We mainly are interested in the fractional operators, the simplest example is given by the fractional Laplace operator $(-\Delta)^s$, for $s \in (0,1)$. This operator is classically defined on the Schwartz space as $\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}\varphi)$, where $\mathcal{F}$ denotes the Fourier transform. This definition some time doesn't permit to immediately understand some properties that functions satisfying some equations involving the fractional Laplace operator have.

For instance, if $u \geq 0$ in $\mathbb{R}^n$ and $(-\Delta)^s u = 0$ in $\Omega \subset \mathbb{R}^n$ for some $s \in (0,1)$, then u satisfies the Harnack inequality. This fact means that there exists a positive constant C such that for every ball of radius r and $B_{4r} \subset \Omega$ then

$$\sup_{B_r} u \leq C \inf_{B_r} u.$$

Using the characterization given in [3], for instance in the case $s = \frac{1}{2}$, the proof is reduced to prove the Harnack inequality for a harmonic function $\tilde{U}$ defined in all the space $\mathbb{R}^n \times \mathbb{R}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ such that $\frac{\partial U}{\partial y}(x,0) = 0$ in Ω. In fact, it results that $(-\Delta)^{1/2}u(x) = \frac{\partial U}{\partial y}(x,0)$, where U is the harmonic function in $\mathbb{R}^n \times (0,+\infty)$ such that $U(x,0) = u$ in $\mathbb{R}^n$. As a consequence, defining $\tilde{U}$ by even reflexion along the plane $y = 0$, $\tilde{U}$ is harmonic in all of $\mathbb{R}^{n+1}$ and the classical theory applies to $\tilde{U}$. As a byproduct, the Harnack inequality holds even for u, see [3]. The definition of fractional Laplace operator of order $s \in (0,1)$ can be stated even as

$$(1) \qquad (-\Delta)^s u(x) = \lim_{\varepsilon \to 0^+} c_{n,s} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

for function continuous function u such that $\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < +\infty$, where $c_{n,s}$ is a normalizing constant such that $-(-\Delta)^s u \to \Delta u$ for $s \to 1^-$ and $-(-\Delta)^s u \to u$ as $s \to 0^+$, see e.g. [7].

The non-locality of these operators is testified not only by the fact that $(-\Delta)^s$ acts via an integral, but even in remarking, for instance, that the Harnack inequality for the $(-\Delta)^{1/2}$, if we do not assume that u is positive defined in all of $\mathbb{R}^n$ but we simply suppose that u is positive on Ω only, then a counter example can be construct, see [17]. On the contrary, the classical Harnack inequality for a harmonic function defined in all of $\mathbb{R}^n$ holds even on every subset Ω of $\mathbb{R}^n$ on which u is positive even if u were negative outside Ω. The definition (1) is the simplest case among a huge amount of cases referring to the hypersingular theory, [20], where different type of quotients are considered. This theory comes from the Marchaud approach, [18]. In the case of the fractional Laplace operators usually it becomes useful to follow the realization obtained with centered increments or following the approach described in [19].

Concerning the definition of the fractional operator, in [8] has been introduced a new definition that permits to take in account functions that grow more than linearly. This fact cannot be admitted using one of the classical definitions of fractional Laplace operator. Nevertheless, it is possible to introduce, see [8], a definition in which when the function has a polynomial growth at infinite, then its fractional Laplacian is just a function, but an equivalence class of functions modulo polynomials of a fixed order.

About the definition of fractional Laplace operators in non-commutative structures, in [12], [11] and [13] have been obtained some achievements about a useful way of defining fractional Laplace operators and further fractional notions like that one of fractional perimeter, for instance in Carnot groups. However this presentation appears in some cases too much complicated by the fact that is introduced via the heat kernel associated with sub-Laplacian operators in Carnot groups.

1.3. **Targets.** The aim of this project is to explore the existence of some results analogous to those obtained in [8], but adapted to more general kernels with respect to the classic ones given by $||h||^{-n-2s}$ in $\mathbb{R}^n$. In particular, we would like to understand if it is possible to select some polynomials, adapted to the selected kernels, characterizing that framework and if the regularity results obtained in [8] might be extended to those operators associated with those new kernels.

In addition, we would like to investigate the chance to extend that approach even in a non-commutative framework and/or to nonlocal operators of higher order as well. In order to achieve these results, it is important to understand what type of changes have to be done in the non-commutative setting in considering some kernels associated with the fundamental solution of the heat operator, [12] and [6]. From this point of view. As a byproduct, a deeper knowledge about the behavior of the kernels in the non-commutative case would be preparatory to face other type of problems in that setting.

For instance, starting from a variational definition of solution, see [5], stated for nonlocal operators in non-commutative structures it would be possible to face nonlocal free boundary problems in that framework.

1.4. **Scheduled Activities.** The hired researcher will have to interplay even with me and some other colleagues of the department. In particular with Eugenio Vecchi and Francesca Colasuonno. Among people that now belong to foreigner institutions, we point out, in particular, Nicola Abatangelo now at the Goethe-Universität at Frankfurt am Main. In addition, it has been scheduled some activities, possibly to be held in a remote way, involving some specialists in the study of nonlocal operators like Enrico Valdinoci and Serana Diperro, at University of Western Australia, and possibly even with Daniela De Silva and Ovidiu Savin at the Columbia University of New York. This last part of the project will be developed in the framework of some visits scheduled in the project: *The interplay of Geometry, Combinatorics and Representation Theory* that is parto of *Bando strutture, "Promozione di iniziative innovative nell'ambito degli accordi quadro" 2020* at the University of Bologna.

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